

Borel Cantelli Lemmas and Extreme Value Theory for Geometric Lorenz Models

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Abstract

We establish dynamical Borel-Cantelli lemmas for nested balls and rectangles centered at generic points in the setting of geometric Lorenz maps. We also establish extreme value statistics for observations maximized at generic points for geometric Lorenz maps and the associated flow.

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1 Introduction

In a chaotic system, the future behavior of the system is very sensitive to the initial conditions and so a statistical description of the system's behavior is often the most appropriate. We may investigate whether suitable versions of classical limit theorems from probability theory such as law of large numbers, central limit theorem, Borel-Cantelli lemma, extreme value theory and so on, hold and use this knowledge to make predictions about the system's behavior. In this paper, we study a particular system, which is Lorenz system, and establish Strong Borel Cantelli lemma and Extreme Value Laws for it.

The equations defining the Lorenz system were first published in the *Journal of Atmospheric Sciences*([11]) as a parametrized polynomial system of differential equations:

$$\begin{aligned}\dot{x} &= \sigma(y - x) \\ \dot{y} &= x(\rho - z) - y \\ \dot{z} &= xy - \beta z\end{aligned}$$

where $\sigma = 10$, $\rho = 28$, $\beta = 8/3$. The system was proposed as a simplified model for thermal fluid convection, motivated by a desire to understand weather systems. What is interesting is that the equations are deterministic but they produce chaotic behavior, with trajectories spiraling around two attractors seemingly randomly. In order to achieve insights on this

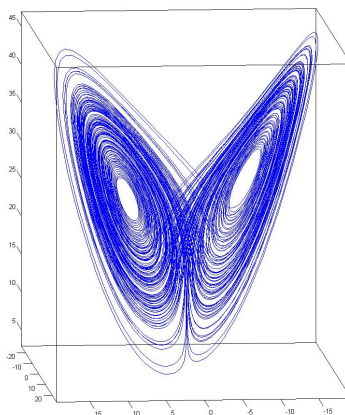


Figure 1: Lorenz attractor

system, a very successful approach was taken by Afraimovich, Bykov and Shil'nikov[1], and

Guckenheimer, Williams[7], independently: they constructed the so-called *Geometric Lorenz models*. These models are flows in three-dimension which have properties very similar to the Lorenz systems and are easier to study. One can rigorously prove the existence of an attractor that contains an equilibrium point of the flow, together with regular solutions. The original proof of the existence of a chaotic attractor was made by Warwick Tucker in the year 2000, with the help of computer (see [16, 17]).

Here we give a brief version of construction of the Geometric Lorenz model, and more detailed version can be found in [6, section 2.1]. Consider a linear system in $[-1, 1]^3$:

$$(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$$

with $\lambda_1, \lambda_2, \lambda_3$ satisfying

$$0 < \frac{\lambda_1}{2} \leq -\lambda_3 < \lambda_1 < -\lambda_2$$

For any initial point $(a, b, c) \in \mathbb{R}^3$ near the equilibrium $(0, 0, 0)$, the trajectories are given by

$$\tilde{L}_t(a, b, c) = (ae^{\lambda_1 t}, be^{\lambda_2 t}, ce^{\lambda_3 t})$$

where \tilde{L}_t denotes the linear flow.

Consider $\Omega = \{(x, y, 1) : |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\} = \Omega^- \cup \Omega^o \cup \Omega^+$, where

$$\Omega^- = \{(x, y, 1) \in \Omega : x < 0\}$$

$$\Omega^+ = \{(x, y, 1) \in \Omega : x > 0\}$$

$$\Omega^o = \{(x, y, 1) \in \Omega : x = 0\}$$

Ω is a transverse section to the linear flow \tilde{L}_t , and since $\lambda_3 < 0$, every trajectory, that would cross Ω , will cross in the direction of the negative z axis. Let $\Omega^* = \Omega^- \cup \Omega^+$ and let $\tilde{\Omega} = \{(x, y, z) : |x| = 1\} = \tilde{\Omega}^- \cup \tilde{\Omega}^+$ with $\tilde{\Omega}^\pm = \{(x, y, z) : x = \pm 1\}$. For each $(a, b, 1) \in \Omega^*$, the time t such that $\tilde{L}_t(a, b, 1) \in \tilde{\Omega}$ is given by

$$|ae^{\lambda_1 t}| = 1 \implies t(a) = -\frac{1}{\lambda_1} \log |a|$$

the time only depends on the first component of the point in Ω^* and $t(a) \rightarrow \infty$ as $a \rightarrow 0$. Thus, we can express the point in $\tilde{\Omega}$ mapped from point $(a, b, 1) \in \Omega^*$ explicitly:

$$\tilde{L}_{t(a)}(a, b, 1) = (\text{sgn}(a), be^{\lambda_2 t(a)}, e^{\lambda_3 t(a)}) = (\text{sgn}(a), b|a|^{-\frac{\lambda_2}{\lambda_1}}, |a|^{-\frac{\lambda_3}{\lambda_1}})$$

where $\text{sgn}(a) = a/|a|$ for $a \neq 0$. In this way, we just defined a map $L : \Omega^* \rightarrow \tilde{\Omega}^\pm$ by

$$L(x, y, 1) = (\text{sgn}(x), y|x|^\beta, |x|^\alpha)$$

where $\beta = -\frac{\lambda_2}{\lambda_1}$, $\alpha = -\frac{\lambda_3}{\lambda_1}$ satisfying $\frac{1}{2} < \alpha < 1 < \beta$, since $0 < \frac{\lambda_1}{2} \leq -\lambda_3 < \lambda_1 < -\lambda_2$.

Then we should let the sets $L(\Omega^*)$ return to the cross section Ω through a flow defined by a suitable composition of a rotation R_{\pm} , an expansion $E_{\pm\theta}$ and a translation \mathbb{T}_{\pm} . More precisely, for $(x, y, z) \in \tilde{\Omega}^{\pm}$,

$$R_{\pm}(x, y, z) = \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 1 & 0 \\ \pm 1 & 0 & 0 \end{pmatrix}$$

and for $(x, y, z) \in \Omega$,

$$E_{\pm\theta}(x, y, z) = \begin{pmatrix} \theta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

θ and \mathbb{T}_{\pm} shall be chosen to satisfy certain conditions.

So the Poincaré first return map, i.e. our Lorenz map, $F : \Omega^* \rightarrow \Omega$, is defined as

$$F(x, y) = \begin{cases} \mathbb{T}_+ \circ E_{+\theta} \circ R_+ \circ L(x, y, 1) & \text{for } x > 0 \\ \mathbb{T}_- \circ E_{-\theta} \circ R_- \circ L(x, y, 1) & \text{for } x < 0 \end{cases}$$

Combining the effect of the rotation with expansion and translation, F must have the form:

$$F(x, y) = (T(x), G(x, y))$$

where $T : I \setminus \{0\} \rightarrow I$ and $G : (I \setminus \{0\}) \times I \rightarrow I$, where $I = [-\frac{1}{2}, \frac{1}{2}]$. Here T is given by

$$T(x) = \begin{cases} f_1(x^\alpha), & x < 0 \\ f_0(x^\alpha), & x > 0 \end{cases}$$

with $f_i(x) = (-1)^i \theta \cdot x + b_i, i \in \{0, 1\}$ such that $\theta \cdot (\frac{1}{2})^\alpha < 1$ and $\theta \cdot \alpha \cdot 2^{1-\alpha} > 1$. T is the quotient map of F , usually referred to as the Lorenz like map, see Figure 2. It has the

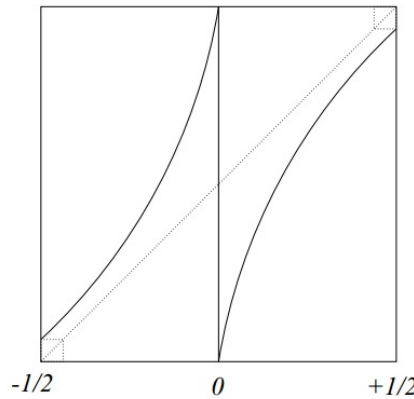


Figure 2: Lorenz like map T

following properties:

1. T is discontinuous at $x = 0$, and the lateral limits $T(0^\pm)$ do exist, $T(0^\pm) = \mp \frac{1}{2}$.
2. T is C^2 on $I \setminus \{0\}$ and $T'(x) > 1$ for all $x \in I \setminus \{0\}$.
3. $\lim_{x \rightarrow 0} T'(x) = +\infty$

And G is given by

$$G(x, y) = \begin{cases} g_1(x^\alpha, y \cdot x^\beta), & x < 0 \\ g_0(x^\alpha, y \cdot x^\beta), & x > 0 \end{cases}$$

where $g_1|I^- \times I \rightarrow I$ and $g_0|I^+ \times I \rightarrow I$ are suitable affine maps. Here $I^- = (-1/2, 0)$, $I^+ = (1/2, 0)$.

Much recent work has focused on the ergodic and statistical properties of Lorenz like maps including rates of mixing, extreme value theory and return time statistics. S. Galatolo and M.J. Pacifico [6] proved that the Poincaré map, i.e. our Lorenz map F , associated to a Lorenz like flow has exponential decay of correlations with respect to Lipschitz observables and the hitting time statistics satisfies a logarithm law.

1.1 Local dimension

Let (M, d) be a metric space and assume that μ is a Borel probability measure on M . Given $x \in M$, let $B_r(x) = \{y \in M : d(x, y) \leq r\}$ be the ball centered at x with radius r . The *local dimension* of μ at $x \in M$ is defined by

$$d_\mu(x) = \lim_{r \rightarrow 0} \frac{\log \mu(B_r(x))}{\log r}$$

if this limit exists. In this case $\mu(B_r(x)) \sim r^{d_\mu(x)}$.

A result of Afraimovich and Pesin [2, Theorem 9] ensures that for the Lorenz system, the local dimension exists and is constant for μ a.e. point.

1.2 Young Tower Structure and Local Product Structure

A recent paper of Araujo, Melbourne and Varandas [3] used a Young Tower construction to establish that a broad range of geometric Lorenz flows are rapidly mixing. Along the way they showed that μ a.e. $p \in M$ has a local product structure (this follows from their Proposition 2.4). More precisely, μ a.e. p has the property that there exists a $r(p) > 0$ such that for all $r < r(p)$ if $A(r)$ is a square of sidelength $2r$, or a ball of radius r , centered at p , then μ a.e. $q \in A(r)$ has a local unstable manifold and a local stable manifold which fully crosses $A(r)$. Moreover if q_1, q_2 are in $A(r)$ then there is a unique point $z = W_{loc}^u(q_1) \cap W_{loc}^s(q_2) \in A(r)$.

1.3 Short returns for one-dimensional Lorenz-like maps

Gupta, Holland and Nicol [8] established extreme value statistics for Lorenz-like maps. The proofs used a crucial estimate on the measure of points with short returns. In particular, on page 21 they showed that for μ a.e. $p \in I$, and for all sufficiently small $r < r(p)$, if $B_r(p)$ is a ball of radius r based at p , then there are constants $C > 0$, $0 < \alpha < 1$ such that for all $1 \leq j \leq (\log r)^5$, $\mu(B_r \cap T^{-j}B_r) \leq \mu(B_r)e^{-(\log r)^\alpha}$.

1.4 Borel Cantelli Lemma

The classical Borel-Cantelli lemmas are as follows: suppose $(\Omega, \mathcal{B}, \mu)$ is a probability space. Let $\mathbf{1}_A$ be the characteristic function of A , given A is a measurable set of Ω . Then

1. if $(A_n)_{n=0}^\infty$ is a sequence of measurable sets in Ω and $\sum_{n=0}^\infty \mu(A_n) < \infty$, then $\mu(x \in A_n \text{ i.o.}) = 0$
2. if $(A_n)_{n=0}^\infty$ is a sequence of independent sets in Ω and $\sum_{n=0}^\infty \mu(A_n) = \infty$, then for μ a.e. $x \in \Omega$

$$\frac{S_n(x)}{E_n} \rightarrow 1$$

where $S_n(x) = \sum_{j=0}^{n-1} \mathbf{1}_{A_j}(x)$ and $E_n = \sum_{j=0}^{n-1} \mu(A_j)$.

In the dynamical systems setting, $T : \Omega \rightarrow \Omega$ is usually considered to be a measure-preserving transformation of the probability space $(\Omega, \mathcal{B}, \mu)$. Suppose that $(A_n)_{n=0}^\infty$ is a sequence of sets in \mathcal{B} such that $\sum_{n=0}^\infty \mu(A_n) = \infty$. Let $E_n = \sum_{j=0}^{n-1} \mu(A_j)$ and $S_n(x) = \sum_{j=0}^{n-1} \mathbf{1}_{A_j} \circ T^j(x)$. Then we call the sequence (A_n) :

1. a **Borel Cantelli sequence(BC)** if $\mu(x : T^n x \in A_n \text{ i.o.}) = 1$, i.e. $S_n(x)$ is unbounded.
2. a **Strong Borel Cantelli sequence(SBC)** if $\lim_{n \rightarrow \infty} \frac{S_n(x)}{E_n} = 1$, a.s..

Remark 1.0.1. If the sequence $(A_n)_{n=0}^\infty$ are nested balls of radius r_n about a point p in the dynamical system (T, Ω, μ) , then the question of whether $T^i x \in A_i$ infinitely often for μ a.e. x is called the shrinking target problem. For the rest of the paper, we establish strong Borel Cantelli Lemmas and Extreme Value Laws for the Lorenz maps F regards to shrinking target property.

1.5 Extreme Value Laws(EVL)

We consider a dynamical system $(\Omega, \mathcal{B}, \mu, F)$ where F preserves an invariant measure μ . Consider the time series $X_0, X_1, X_2 \dots$ arising from this system by evaluating a given random variable(r.v.) $\varphi : \Omega \rightarrow \mathbb{R} \cup \{\pm\infty\}$ along the orbits of the system, that is to say, we define

$$X_n = \varphi \circ F^n \tag{1.1}$$

for each $n \in \mathbb{N}$. Apparently, X_0, X_1, \dots defined in this way is not an independent sequence, but F -invariance of μ guarantees that the stochastic process is stationary.

Here we suppose that φ has one global maximum at $\zeta \in \Omega$ ($\varphi(\zeta) = +\infty$ is allowed). And let $u_F := \varphi(\zeta)$. By assuming that φ and μ are sufficiently regular, the event

$$U(u) := \{x \in \Omega : \varphi(x) > u\} = \{X_0 > u\}$$

corresponds to a topological ball centered at ζ for u sufficiently close to u_F . Furthermore, $\mu(U(u))$ varies continuously as a function of u on a neighbourhood of u_F .

Definition 1.1. (Logarithmic singularity) Consider a function $\varphi : \Omega \rightarrow \mathbb{R}$ and a point $x_0 \in \Omega$. Let d be a distance function on Ω . We say that φ has a logarithmic singularity at the point x_0 if x_0 has a neighborhood where $\varphi(x) = -C \log d(x, x_0) + g(x)$ with $C > 0$, where g is bounded and has a finite limit as $x \rightarrow x_0$.

We are always interested in studying the extremal behavior of the stochastic process X_0, X_1, \dots , and it is associated with the occurrence of exceedances of high levels u . When u is close to u_F , the occurrence of the event $\{X_j > u\}$ means that the occurrence of an exceedance at time $j \in \mathbb{N}_0$. This is equivalent to saying that $F^j(x) \in U(u)$, i.e. the orbit of the point x hits the ball $U(u)$ at time j .

In order to consider the extremal behavior of the system for which we define a new sequence of random variables M_1, M_2, \dots given by

$$M_n = \max\{X_0, \dots, X_{n-1}\}$$

Definition 1.2. (Extreme Value Laws) We say that we have an *EVL* for M_n if there is a non-degenerate distribution function(d.f.) $G : \mathbb{R} \rightarrow [0, 1]$ with $G(0) = 0$ and, for every $v > 0$, there exists a sequence of levels $u_n = u_n(v)$, $n = 1, 2, \dots$, such that

$$\lim_{n \rightarrow \infty} n\mu(X_0 > u_n) = v \tag{1.2}$$

and for which the following holds:

$$\mu(M_n \leq u_n) \rightarrow \bar{G}(v) \text{ as } n \rightarrow \infty$$

where $\bar{G} = 1 - G$.

The motivation for using such normalising sequences (1.2) comes from the case when X_0, X_1, \dots are independent and identically distributed (i.i.d.). In this i.i.d. setting, it is clear that $P(M_n \leq u) = (Z(u))^n$, where Z is the distribution function of X_0 , i.e. $Z(x) := P(X_0 \leq x)$. Therefore, condition (1.2) indicates that

$$P(M_n \leq u_n) = (1 - P(X_0 > u_n))^n \sim \left(1 - \frac{v}{n}\right)^n \rightarrow e^{-v},$$

as $n \rightarrow \infty$. This implies that, approximately, the waiting times between exceedances of u_n is exponentially distributed. Moreover, the reciprocal is also true. Note that in this case $G(v) = 1 - e^{-v}$ is the standard exponential d.f..

Remark 1.2.1. We will give result on Lorenz flows f_t , in which case we consider continuous time stochastic process $\{X_t\}$ and define the process of successive maxima $M_T := \sup_{0 \leq t \leq T} \{X_t\}$.

Remark 1.2.2. For independent and identically distributed(i.i.d.) processes, there are only three possible types of non-degenerate extremal distributions (subject to linear scaling):

- Type I

$$G(x) = e^{-e^{-x}}, -\infty < x < \infty$$

- Type II

$$G(x) = \begin{cases} 0 & \text{if } x < 0; \\ e^{-x^{-\alpha}} & \text{for some } \alpha \text{ if } x > 0. \end{cases}$$

- Type III

$$G(x) = \begin{cases} e^{-(-x)^\alpha} & \text{for some } \alpha > 0 \text{ if } x < 0; \\ 1 & \text{if } x > 0. \end{cases}$$

For dependent stationary processes $\{X_n\}$, *EVL* will satisfy under $D_3(u_n)$ and $D'(u_n)$, which will be introduced in the next subsection.

For μ a.e. x_0 , if we consider

$$\varphi(x) = -\log d(x, x_0) \tag{1.3}$$

where $d(\cdot, \cdot)$ is the local metric on Ω and x_0 is a fixed point. And define $U_n = \{X_0 > u_n\}$, where $u_n = u_n(v)$ such that $\mu(U_n) = e^{-v}/n$. Here u_n is an increasing sequence going to $\varphi(x_0)$ (which is $+\infty$) and assume U_n corresponds to a topological ball centered at x_0 with radius e^{-u_n} . Then the corresponding processes $\{X_n\}$ will satisfy Type I extremal distribution.

To be consistent, if not specified, we will use the definition and assumption above for section on EVL.

Remark 1.2.3. We define a function that we refer to as *first hitting time function* to a set $A \in \mathcal{B}$, and denote by $r_A : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ where

$$r_A(x) = \min\{j \in \mathbb{N} \cup \{\infty\} : F^j(x) \in A\}$$

The restriction of r_A to A is called the *first return time function* to A , denoted by $R(A)$, as the minimum of the return time function to A , i.e.

$$R(A) = \min_{x \in A} r_A(x)$$

In [4], the link between *EVL* and *Hitting Time Statistics* (HTS)/ *Returning Time Statistics* (RTS) (for balls) of stochastic processes defined by (1.1) was established and for systems with an absolutely continuous invariant measure, they have been shown to be equivalent to each other, i.e. if such processes have an EVL G then the system has HTS G as well for balls “centered” at ζ and vice versa. So it is natural to use the observable (1.3), it gives balls centered at x_0 . And we have:

$$\{M_n \leq u_n\} = \{r_{\{X_0 > u_n\}} > n\} = U_n^c$$

We may also consider the statistics of multiple returns, which we discuss in the next subsection.

1.5.1 Rare events points processes and respective convergence

Let's introduce some formalism first. Let \mathcal{W} denote the semi-ring of subsets of \mathbb{R}^+ whose elements are half closed half open intervals $[b, c)$, for $b, c \in \mathbb{R}^+$. Let \mathcal{V} be the ring generated by \mathcal{W} . For each element $I \in \mathcal{V}$, there exist $k \in \mathbb{N}$ and k intervals $J_1, \dots, J_k \in \mathcal{W}$ such that $I = \cup_{j=1}^k J_j$. For $J = [b, c) \in \mathcal{W}$ and $\alpha \in \mathbb{R}$, we denote $\alpha J := [\alpha b, \alpha c)$ and $J + \alpha := [b + \alpha, c + \alpha)$. Similarly, for $I \in \mathcal{V}$, we define $\alpha I := \alpha J_1 \cup \dots \cup \alpha J_k$ and $I + \alpha := (J_1 + \alpha) \cup \dots \cup (J_k + \alpha)$.

Definition 1.3 (Rare Event Point Process). For stationary processes X_0, X_1, \dots and sequences $(u_n)_{n \in \mathbb{N}}$ satisfying (1.2), we define the *Rare Event Point Process*(REPP) by setting:

$$N_n(I) := \sum_{j \in a_n I \cap \mathbb{N}_0} \mathbf{1}_{\{X_j > u_n\}} \quad \text{for every } I \in \mathcal{V}$$

where $a_n I \in \mathcal{V}$, is the re-scaled time period. And a_n is taken to be $1/\mu(X_0 > u_n)$, according to Kac's Theorem, the expected waiting time before the occurrence of one exceedance. In fact, this is counting the number of exceedances during the re-scaled time period $a_n I$.

Definition 1.4 (Poisson process of intensity θ). Let S_1, S_2, \dots be an i.i.d. sequence of random variables with common exponential distribution of mean $1/\theta$. Given this sequence of r.v., for $I \in \mathcal{V}$, set

$$N(I) = \int \mathbf{1}_I d\left(\sum_{i=1}^{\infty} \delta_{S_1 + \dots + S_i}\right)$$

where δ_l denotes the Dirac measure at $l > 0$ and, as before, $I \in \mathcal{V}$. We call such N is a *Poisson process of intensity θ* . In special case, when $I = [0, t)$, we also denote $N([0, t))$ by $N(t)$.

Remark 1.4.1. If $\theta = 1$ then N is called a standard Poisson process and, for every $t > 0$, the random variable $N(t)$ has a Poisson distribution of mean t .

In [5, page 4], two conditions $D_3(u_n)$ and $D'(u_n)$ are given on the dependence structure of a general stationary stochastic process to ensure that the REPP N_n converges in distribution to a standard Poisson process. Also they ensure dependent stationary stochastic processes to have EVL. These two conditions are the following:

Condition ($D_3(u_n)$). We say that $D_3(u_n)$ holds for the sequence X_0, X_1, X_2, \dots if for any integers l, t and n

$$\left| \mu(\{X_0 > u_n\} \cap \{M_{t,l} \leq u_n\}) - \mu(\{X_0 > u_n\})\mu(\{M_l \leq u_n\}) \right| \leq \gamma(n, t)$$

where $M_{t,l} = \max\{X_t, X_{t+1}, \dots, X_{t+l-1}\}$, and $\gamma(n, t)$ is nonincreasing in t for each n and $n\gamma(n, t_n) \rightarrow 0$ as $n \rightarrow \infty$ for some sequence $t_n = o(n)$, $t_n \rightarrow \infty$.

Condition ($D'(u_n)$): The condition $D'(u_n)$ is said to hold for the stationary sequence $\{X_i\}$ and the sequence $\{u_n\}$ if

$$\limsup_{n \rightarrow \infty} n \sum_{j=1}^{[n/k]} \mu(X_0 > u_n, X_j > u_n) \rightarrow 0$$

as $k \rightarrow \infty$.

While $D_3(u_n)$ is a condition on the long range dependence structure of the stochastic process X_0, X_1, \dots , condition $D'(u_n)$ is a non-clustering condition, which states that if a large reading is observed (say the level u_n) at some time $j < n$, then one must wait for a large time $o(n) \rightarrow \infty$ before another reading larger than or equal to u_n is observed.

Assumption A: For μ a.e. $p \in \Omega$ there exists $\tilde{d} = \tilde{d}(p) > 0$ such that if $A_{r,\epsilon}(p) = \{y \in \Omega : r \leq d(p, y) \leq r + \epsilon\}$ is a shell of inner radius r and outer radius $r + \epsilon$ about the point p , and if r is sufficient small and $0 < \epsilon \ll r < 1$, then $\mu(A_{r,\epsilon}) < \epsilon^{\tilde{d}}$.

Remark 1.4.2. For systems satisfying Assumption A, condition $D_3(u_n)$ often follows easily if there are good enough estimates on decay of correlation for observations in a suitable Banach space.

J. M. Freitas, N. Haydn, and M. Nicol [5] states that the REPP N_n converges in distribution to a standard Poisson process for functions maximized at generic points in a variety of billiard systems. They prove this by verifying that the conditions $D_3(u_n)$ and $D'(u_n)$ hold for such systems.

1.6 Main results

In section 2, to introduce the main ideas of our analysis of the Lorenz system in a simpler setting, we establish results of independent interest, namely shrinking target properties for general skew product maps which do preserve the two-dimensional Lebesgue measure. We have the following theorem:

Theorem 1.5. *Suppose $(\Omega, \mathcal{B}, m_2)$ is a probability space, where $\Omega = I \times I$, with $I = [-\frac{1}{2}, \frac{1}{2}]$, and m_2 is the two-dimensional Lebesgue measure. Let m denote the one-dimensional Lebesgue measure. Suppose $F : \Omega \rightarrow \Omega$ is a map in the form $F(x, y) = (T(x), G(x, y))$, where $T : I \rightarrow I$. Here F preserves the two-dimensional Lebesgue measure and T preserves the one-dimensional Lebesgue measure. In addition, T satisfies exponential decay of correlation with observables in BV norm versus L^1 norm, i.e.*

$$|\int \phi \psi \circ T^n dm - \int \phi dm \int \psi dm| \leq C \theta^n \|\phi\|_{BV} \|\psi\|_1$$

and F satisfies exponential decay of correlation with observables in Lipschitz norm versus Lipschitz norm:

$$|\int \phi \psi \circ F^n dm_2 - \int \phi dm_2 \int \psi dm_2| \leq C \alpha^n \|\phi\|_{Lip} \|\psi\|_{Lip}$$

Consider nested balls $(B_i(p))$, centered at some $p \in \Omega$, with $m_2(B_i(p)) \geq \frac{C}{i^{\gamma_1}}$ for some $\gamma_1 > 0$, and $\limsup(\log i)(m_2(B_i))^{\frac{1}{2}} \leq C$. Then we have the Strong Borel Cantelli property

$$\frac{S_n(x, y)}{E_n} \rightarrow 1 \text{ a.s.}$$

where $S_n(x, y) = \sum_{j=0}^{n-1} 1_{B_j} \circ F^j(x, y)$, $E_n = \sum_{j=0}^{n-1} m_2(B_j)$.

Then in section 3, we establish the shrinking target property for the two-dimensional Lorenz map F and Extreme Value Laws for the system with the Lorenz map. When we consider shrinking target balls, they have different shapes according to different metrics. Technically, balls of different shapes are equivalent to each other, but we are able to deal with rectangle balls and circle balls. While rectangle balls consist of local unstable manifold of same length, circle balls don't. For rectangle balls, we prove the following theorem:

Theorem 1.6. *Consider a sequence of nested square balls A_i , centered at a point p , of side length $2r(i)$ such that $\mu(A_i) \geq \frac{C_2}{i^{\gamma_1}}$, with $\gamma_1 \geq 0$. Assume that p has a local product structure for sufficiently small neighborhoods. Also we assume $(\log i)m(\pi(A_i)_\gamma)$ is bounded, where γ is any local unstable manifold of A_i and π is the projection map onto the first dimension. Then if $\sum_i \mu(A_i)$ diverges we have the Strong Borel Cantelli Property for the squares A_i of sidelength $2r(i)$.*

And for circle balls, we show:

Theorem 1.7. *Consider a sequence of nested circle balls A_i , centered at a point p , of radius $r(i) = e^{-u_i}$. And assume $\mu(A_i) \geq \frac{C_2}{i^{\gamma_1}}$, with $\gamma_1 \geq 0$. Also assume that p has a local product structure for sufficiently small neighborhoods. Then if $\sum_i \mu(A_i)$ diverges we have the Strong Borel Property for the balls A_i .*

We use techniques from subsection 3.2(About EVL) to prove Theorem 1.7, therefore we will prove it after we establish Extreme Value Laws for the system with the Lorenz map F . We also show:

Theorem 1.8. *Consider the dynamical systems $(\Omega, \mathcal{B}, \mu, F)$, where F is the Lorenz map and F preserves the measure μ , whose decomposition on the unstable leaves is absolutely continuous with respect to the one-dimensional Lebesgue measure. If φ is defined as (1.3), and let $X_n = \varphi \circ F^n$. We assume x_0 is not periodic under F . Then X_n satisfies a Type I extreme value law, i.e.*

$$\lim_{n \rightarrow \infty} \mu(M_n \leq u_n) = e^{-e^{-v}}$$

In section 4, we extend our results to the Lorenz flow.

2 Volume Preserving Skew Products

As we mentioned in the introduction, the Lorenz map F has a skew product form $F(x, y) = (T(x), G(x, y))$. And F does not preserve the two-dimensional Lebesgue measure m_2 but the

one-dimensional map T preserves a measure absolutely continuous with respect to the one-dimensional Lebesgue measure m , with a Lipschitz density. In this section, we talk about general skew product maps $\tilde{F}(x, y) = (T(x), G(x, y))$, which do preserve the two-dimensional Lebesgue measure and T does preserve the one-dimensional Lebesgue measure. We stated our Theorem 1.5 in the introduction, now let's prove it.

Proof of Theorem 1.5.

Let $f_k = \mathbf{1}_{B_k} \circ F^k(x, y)$, $E(f_k) = m_2(B_k)$ and let $a \geq \frac{-7\gamma_1}{\log \alpha}$; we will use a later. To prove Strong Borel Cantelli Property, according to [9], it suffices to show the (SP) property, i.e. for all $m < n$

$$\sum_{i=m}^n \sum_{j=i+1}^n (E(f_i f_j) - E(f_i)E(f_j)) \leq C \sum_{i=m}^n E(f_i)$$

We calculate

$$\begin{aligned} E(f_i f_j) &= \int \mathbf{1}_{B_i} \circ F^i(x, y) \cdot \mathbf{1}_{B_j} \circ F^j(x, y) dm_2 \\ &= \int \mathbf{1}_{B_i} \cdot \mathbf{1}_{B_j} \circ F^{j-i}(x, y) dm_2 \\ &= m_2(B_i \cap F^{-(j-i)} B_j) \\ &\leq C_1(m_2(B_i))^{\frac{1}{2}} \cdot m(\pi_X B_i \cap T^{-(j-i)} \pi_X B_j)^1 \\ &= C_1(m_2(B_i))^{\frac{1}{2}} \cdot \int \mathbf{1}_{\pi_X B_i} \cdot \mathbf{1}_{\pi_X B_j} \circ T^{j-i}(x) dm \\ &\leq C_1(m_2(B_i))^{\frac{1}{2}} \cdot \left(\int \mathbf{1}_{\pi_X B_i} \int \mathbf{1}_{\pi_X B_j} + C\theta^{j-i} \|\mathbf{1}_{\pi_X B_i}\|_{BV} \|\mathbf{1}_{\pi_X B_j}\|_1 \right) \\ &\leq C_1(m_2(B_i))^{\frac{1}{2}} \cdot \left[(m_2(B_i))^{\frac{1}{2}} \cdot (m_2(B_j))^{\frac{1}{2}} + C\theta^{j-i} (m_2(B_j))^{\frac{1}{2}} \right] \\ &\leq C_1(m_2(B_i))^{\frac{3}{2}} + C\theta^{j-i} m_2(B_i) \end{aligned}$$

since $\int \mathbf{1}_{\pi_X B_i} dm = m(\pi_X B_i) = (m_2(B_i))^{\frac{1}{2}}$.

¹Here $m_2(B_i \cap F^{-(j-i)} B_j) \leq C_1(m_2(B_i))^{\frac{1}{2}} \cdot m(\pi_X B_i \cap T^{-(j-i)} \pi_X B_j)$ is because that if $\tilde{x} = (x, y) \in B_i$ and $F^{j-i}(\tilde{x}) \in B_j$, then their projection $x \in \pi_X B_i$ and $T^{j-i}(x) \in \pi_X B_j$ so $m_2(B_i \cap F^{-(j-i)} B_j) = \int m((B_i \cap F^{-(j-i)} B_j)_y) dm(y) = \int m(\{\tilde{x} = (x, y) | \tilde{x} \in B_i, F^{j-i}(\tilde{x}) \in B_j\}) dm(y) \leq \int m(x \in \pi_X B_i, T^{j-i}(x) \in \pi_X B_j) m(y)$, where $\pi_X B_i$ is the projection of the maximal horizontal section of the ball B_i .

Recall $a \geq \frac{-7\gamma_1}{\log \alpha}$, so that

$$\begin{aligned}
& \sum_{j=i+1}^n (E(f_i f_j) - E(f_i)E(f_j)) \\
& \leq \left(\sum_{j=i+1}^{i+a \log i} + \sum_{j>i+a \log i} \right) [E(f_i f_j) - E(f_i)E(f_j)] \\
& \leq C_1 (\log i) (m_2(B_i))^{\frac{3}{2}} + C m_2(B_i) + \sum_{j>i+a \log i} C \alpha^{j-i} \|\tilde{f}_i\|_{Lip} \|\tilde{f}_j\|_{Lip} + O\left(\frac{1}{i^{7/2}}\right)
\end{aligned}$$

where \tilde{f}_i is a Lipschitz approximation to f_i , which is constructed as following: $\tilde{f}_i = \mathbf{1}_{B_i}(x)$ if $x \in B_i$, $\tilde{f}_i = 0$ if $d(B_i, x) > 1/i^{3\gamma_1}$, $0 \leq \tilde{f}_i \leq 1$ and $\|\tilde{f}_i\|_{Lip} \leq i^{3\gamma_1}$.

Then

$$\begin{aligned}
\sum_{j>i+a \log i} \alpha^{j-i} \|\tilde{f}_i\|_{Lip} \|\tilde{f}_j\|_{Lip} & \leq \sum_{j>i+a \log i} \alpha^{j-i} i^{3\gamma_1} j^{3\gamma_1} \\
& = \sum_{\beta=1}^{\infty} \alpha^{a \log i + \beta} i^{3\gamma_1} (i + a \log i + \beta)^{3\gamma_1} \\
& \leq \alpha^{a \log i} i^{3\gamma} C i^{3\gamma_1} \\
& \leq \frac{C}{i^{\gamma_1}} \leq C m_2(B_i)
\end{aligned}$$

Since $(\log i)(m_2(B_i))^{\frac{1}{2}} \leq C$, the (SP) property is satisfied. \square

Remark 2.0.1. (SP) property (Sprindzuk Property) is derived from Gal-Koksma theorem, which is given in the Appendix. Once we have the (SP) property, then the Strong Borel Cantelli Property is established because in the Gal-Koksma theorem, if we take $f_k(\omega) = \mathbf{1}_{B_k} \circ F^k(x, y)$, $h_k = g_k = E(f_k) = m_2(B_k)$, dividing both sides of the equation by $\sum_{k=1}^n g_k$, we will have $\frac{S_n(x, y)}{E_n} \rightarrow 1$ a.s..

3 Lorenz System

In this section, we present our results on the statistical properties of the Lorenz system, i.e. Borel Cantelli Lemma and Extreme Value Laws. Recall that the Lorenz map F does not preserve the two-dimensional Lebesgue measure m_2 , but preserves an invariant measure μ which has absolutely continuous conditional measures on local unstable manifolds.

3.1 Borel Cantelli Lemma

We let $A_r(p)$ denote the square of sidelength $2r$ centered at a point p in the two dimensional space $I \times I$. As a consequence of [3, Proposition 2.4], for μ a.e. p , there exists an $r(p) > 0$ such

that for all $r < r(p)$, $A_r(p)$ has a local product structure and in particular μ a.e. $q \in A_r(p)$ has a local unstable manifold $\gamma(q) := W_{loc}^u(q)$ which extends fully across $A_r(p)$. The local stable manifolds are arbitrarily long for μ a.e. q . The set of local unstable manifolds $\Gamma = \{\gamma(q)\}$ partition $A_r(p)$ up to a set of zero μ measure i.e. $\mu(A_r(p)) = \mu(\cup_{q \in A_r(p)} \gamma(q) \cap A_r(p))$. We will drop the dependence on q and write $\Gamma = \{\gamma\}$ for simplicity.

Before we prove Theorem 1.6, we introduce the notation.

Given such a point p we let A be a square based at p , with sidelength smaller than $2r(p)$.

Let $A_\gamma = A \cap \gamma$ for $\gamma \in \Gamma$,

$$\mu(A) = \int_I m_\gamma(A_\gamma) d\nu(\gamma)$$

where m_γ is the induced measure of μ on γ and ν is conditional measure in the decomposition of μ with respect to the partition $\{\gamma\}$.

Let π be the projection map onto the first dimension and note that $m_\gamma(A_\gamma) \sim m(\pi A)$.

For the Lorenz map F (see [6, Theorem 4.7]), we have exponential decay in Lipschitz versus Lipschitz

$$|\int \phi \psi \circ F^n d\mu - \int \phi d\mu \int \psi d\mu| \leq C\alpha^n \|\phi\|_{Lip} \|\psi\|_{Lip}$$

and for the base map T (see [6, Propostion 2.2]), we have exponential decay in L^1 versus BV

$$|\int \phi \psi \circ T^n dm - \int \phi dm \int \psi dm| \leq C\theta^n \|\phi\|_{BV} \|\psi\|_1$$

By taking $\phi = \psi = \mathbf{1}_{\pi A} = \mathbf{1}_{\pi A_\gamma}$, we have

$$m(\pi A_\gamma \cap T^{-n}(\pi A_\gamma)) - (m(\pi A_\gamma))^2 \leq C\theta^n \|\mathbf{1}_{\pi A_\gamma}\|_{BV} \|\mathbf{1}_{\pi A_\gamma}\|_1$$

That is

$$m(\pi A_\gamma \cap T^{-n}(\pi A_\gamma)) \leq (m(\pi A_\gamma))^2 + C'\theta^n m(\pi A_\gamma)$$

since $\|\mathbf{1}_{\pi A_\gamma}\|_{BV}$ is bounded.

Proof of Theorem 1.6.

We will establish (SP) property. Without loss of generality, we assume $i < j$. We notice

$m(\pi(A_i)_\gamma)$ is equal for all $\gamma \in \Gamma$ since they are square balls. Thus,

$$\begin{aligned}
& \mu(A_i \cap F^{-(j-i)} A_j) \leq \mu(A_i \cap F^{-(j-i)} A_i) \\
& \sim \int_I m(\tilde{x} \in (A_i)_\gamma : F^{j-i}(\tilde{x}) \in A_i) d\nu(\gamma) \\
& \leq \int_I m(\pi(A_i)_\gamma \cap T^{-(j-i)}(\pi(A_i)_\gamma)) d\nu(\gamma) \\
& \leq \int_I (m(\pi(A_i)_\gamma))^2 + C' \theta^{(j-i)} m(\pi(A_i)_\gamma) d\nu(\gamma) \\
& = \int_I (m(\pi(A_i)_\gamma))^2 d\nu(\gamma) + C' \theta^{(j-i)} \int_I m(\pi(A_i)_\gamma) d\nu(\gamma) \\
& = \int_I (m(\pi(A_i)_\gamma))^2 d\nu(\gamma) + C' \theta^{(j-i)} \mu(A_i) \\
& \leq C m(\pi(A_i)_\gamma) \int_I m(\pi(A_i)_\gamma) d\nu(\gamma) + C' \theta^{(j-i)} \mu(A_i) \\
& = C m(\pi(A_i)_\gamma) \mu(A_i) + C' \theta^{(j-i)} \mu(A_i)
\end{aligned}$$

Since $\mathbf{1}_{A_n}$ is not Lipschitz, we let ϕ_n be a Lipschitz approximation of $\mathbf{1}_{A_n}$ such that

1. $\|\mathbf{1}_{A_n} - \phi_n\|_1 < (\mu(A_n))^3$
2. $\|\phi_n\|_{Lip} < (\mu(A_n))^{-3}$

Then

$$\begin{aligned}
& \left| \int \mathbf{1}_{A_i} \cdot \mathbf{1}_{A_j} \circ F^{j-i} d\mu - \int \mathbf{1}_{A_i} d\mu \int \mathbf{1}_{A_j} d\mu \right| \\
& = \left| \int ([\mathbf{1}_{A_i} - \phi_i] + \phi_i) \cdot ([\mathbf{1}_{A_j} - \phi_j] + \phi_j) \circ F^{j-i} d\mu - \int ([\mathbf{1}_{A_i} - \phi_i] + \phi_i) d\mu \int ([\mathbf{1}_{A_j} - \phi_j] + \phi_j) d\mu \right| \\
& \leq \left| \int \phi_i \phi_j \circ F^{j-i} d\mu - \int \phi_i d\mu \int \phi_j d\mu \right| + \left| \int (\mathbf{1}_{A_i} - \phi_i)(\mathbf{1}_{A_j} - \phi_j) \circ F^{j-i} d\mu \right| \\
& \quad + \left| \int \phi_i(\mathbf{1}_{A_j} - \phi_j) \circ F^{j-i} d\mu \right| + \left| \int (\mathbf{1}_{A_i} - \phi_i) \phi_j \circ F^{j-i} d\mu \right| \\
& \quad + \left| \int (\mathbf{1}_{A_i} - \phi_i) d\mu \int (\mathbf{1}_{A_j} - \phi_j) d\mu \right| + \left| \int \phi_i d\mu \int (\mathbf{1}_{A_j} - \phi_j) d\mu \right| \\
& \quad + \left| \int (\mathbf{1}_{A_i} - \phi_i) d\mu \int \phi_j d\mu \right| \\
& \leq C \alpha^{j-i} \|\phi_i\|_{Lip} \|\phi_j\|_{Lip} + \tilde{C}(\mu(A_i))^3
\end{aligned}$$

If we choose $a \geq \frac{-7\gamma_1}{\log \alpha}$, then

$$\begin{aligned}
\sum_{j>i+a \log i} \alpha^{j-i} \|\phi_i\|_{Lip} \|\phi_j\|_{Lip} &\leq \sum_{j>i+a \log i} \alpha^{j-i} i^{3\gamma_1} j^{3\gamma_1} \\
&= \sum_{\beta=1}^{\infty} \alpha^{a \log i + \beta} i^{3\gamma_1} (i + a \log i + \beta)^{3\gamma_1} \\
&\leq \alpha^{a \log i} i^{3\gamma_1} C i^{3\gamma_1} \\
&\leq \frac{C}{i^{\gamma_1}} \\
&\leq C \mu(A_i)
\end{aligned}$$

Thus, let $f_k = \mathbf{1}_{A_k} \circ F^k(x, y)$, $E(f_k) = \mu(A_k)$, so that we have

$$\begin{aligned}
&\sum_{j=i+1}^n (E(f_i f_j) - E(f_i) E(f_j)) \\
&= \sum_{j=i+1}^n \mu(A_i \cap F^{j-i} A_j) - \mu(A_i) \mu(A_j) \\
&= \sum_{j=i+1}^{i+a \log i} [\mu(A_i \cap F^{j-i} A_j) - \mu(A_i) \mu(A_j)] + \sum_{j>i+a \log i} [\mu(A_i \cap F^{j-i} A_j) - \mu(A_i) \mu(A_j)] \\
&\leq \sum_{j=i+1}^{i+a \log i} [m(\pi(A_i)_\gamma) \mu(A_i) + C' \theta^{j-i} \mu(A_i)] + C \mu(A_i) \\
&\leq C(\log i) m(\pi(A_i)_\gamma) \mu(A_i) + C_1 \mu(A_i) + C_2 \mu(A_i) \leq \tilde{C} \mu(A_i)
\end{aligned}$$

We have established the (SP) property and thus the strong Borel Cantelli lemma for $\{A_i\}$. □

For more general case, i.e. circle balls, we have Theorem 1.7. As we mentioned in the introduction, the proof of Theorem 1.7 needs the techniques from the subsection on EVL so we will do the proof then.

3.2 Extreme Value Laws

In this section, we establish EVL for Lorenz maps by essentially showing that the two conditions $D_3(u_n)$ and $D'(u_n)$, which were introduced in section 1.5, are satisfied. Recall:

Condition ($D_3(u_n)$). We say that $D_3(u_n)$ holds for the sequence X_0, X_1, X_2, \dots if for any integers l, t and n

$$\left| \mu(\{X_0 > u_n\} \cap \{M_{t,l} \leq u_n\}) - \mu(\{X_0 > u_n\}) \mu(\{M_l \leq u_n\}) \right| \leq \gamma(n, t)$$

where $M_{t,l} = \max\{X_t, X_{t+1}, \dots, X_{t+l-1}\}$, and $\gamma(n, t)$ is nonincreasing in t for each n and $n\gamma(n, t_n) \rightarrow 0$ as $n \rightarrow \infty$ for some sequence $t_n = o(n)$.

Condition ($D'(u_n)$): The condition $D'(u_n)$ is said to hold for the stationary sequence $\{X_i\}$ and the sequence $\{u_n\}$ if

$$\limsup_{n \rightarrow \infty} n \sum_{j=1}^{[n/k]} \mu(X_0 > u_n, X_j > u_n) \rightarrow 0$$

as $k \rightarrow \infty$.

Before we prove Theorem 1.8, let's prove the following two lemmas:

Lemma 1. *Suppose we have a local product structure about a point x_0 and the local dimension exists, denoted by d . Then Assumption A is satisfied.*

Proof. As before, the conditional measure μ_γ is equivalent to Lebesgue in the local unstable direction, and r is small, i.e., $r < 1$. Let $\epsilon = r^w$, with $w > 1$. We need to prove that the measure of the annular region $S = A_{r+\epsilon}(x_0)/A_r(x_0)$ is small.

We decompose μ in a neighborhood of x_0 as follows

$$\mu(A) = \int_{\gamma \in \Gamma} m(\gamma \cap A) d\nu(\gamma)$$

where γ is the foliation into local unstable manifolds. Since we have a local product structure at x_0 , these extend all the way across a sufficiently small rectangular neighborhood of x_0 .

Now consider the equation of the circles $x^2 + y^2 = r^2$ and $x^2 + y^2 = (r + \epsilon)^2 = r^2 + 2r^{w+1} + r^{2w}$. The larger circle contains some local unstable manifolds which are not in the smaller circle but the greatest length of these is found by setting $y^2 = r^2$ in the second equation and solving for $\delta x \leq r^{\frac{w+1}{2}}$. Their length is less than $r^{\frac{w+1}{2}}$, so that

$$\mu(S) \leq \int_{\Gamma} (S \cap \gamma) d\nu(\gamma) < r^{\frac{w+1}{2}} < \epsilon^{\frac{w+1}{2w}} < \epsilon^{1/2}$$

□

Lemma 2. (a) *For μ a.e. x_0 , for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n \geq N$*

$$\frac{1}{d + \epsilon}(v + \log n) \leq u_n(v) \leq \frac{1}{d - \epsilon}(v + \log n)$$

where d is the local dimension.

(b) *Denote by $S(n, x_0) = A_{e^{-u_n}}(x_0)/A_{e^{-u_n} - e^{-u_n^2}}(x_0)$, the annulus region between balls centered at x_0 of radius e^{-u_n} and $e^{-u_n} - e^{-u_n^2}$. There exists $\delta = \delta(x_0) \in (0, 1)$ such that for n large enough*

$$\mu(S(n, x_0)) \leq C_3 n^{-2\delta v - \delta \log n}$$

Proof. (a) By the definition of the local dimension, for any $\epsilon > 0$, there is an N such that for all $n \geq N$, $(e^{-u_n})^{(d+\epsilon)} \leq \mu(U_n) \leq (e^{-u_n})^{(d-\epsilon)}$, and $\mu(U_n) = e^{-v}/n$. We get immediately

$$\frac{1}{d+\epsilon}(v + \log n) \leq u_n(v) \leq \frac{1}{d-\epsilon}(v + \log n)$$

(b) According to Subsection 1.1 and 1.2, Lorenz system has local dimension and local structure, and by Lemma 1, Assumption A is satisfied for Lorenz system. So there exists a $\delta \in (0, 1)$ such that

$$\begin{aligned} \mu(S(n, x_0)) &\leq C(e^{-(u_n^2)})^\delta \\ &= Ce^{-(u_n^2)\delta} \\ &\leq C \exp\left(-\frac{\delta}{(d+\epsilon)^2}(v + \log n)^2\right) \\ &\leq C_3 n^{-2\delta'v - \delta' \log n} \end{aligned}$$

□

Proof of Theorem 1.8.

To show *EVL* for the Lorenz system, it suffices to show $D_3(u_n)$ and $D'(u_n)$. Then, as mentioned in Remark 1.4.2, $D_3(u_n)$ is easily proven if the system satisfies Assumption A and good enough estimates for decay of correlations. We already know Assumption A is satisfied for Lorenz systems, and we have exponential decay of correlation for Lorenz map

$$\left| \int \phi \psi \circ F^n d\mu - \int \phi d\mu \int \psi d\mu \right| \leq C \alpha^n \|\phi\|_{Lip} \|\psi\|_{Lip}$$

So we prove $D_3(u_n)$ in the following paragraph.

By part (b) of Lemma 2, we have $\mu(S(n, x_0)) \leq C_3 n^{-2\delta v - \delta \log n}$, where $\delta = \delta(x_0) \in (0, 1)$ for n large enough and v could be any number. Take ϕ_n to be the Lipschitz approximation of $\mathbf{1}_{U_n} = \mathbf{1}_{\{X_0 > u_n\}}$ such that $\phi_n(x) = 1$ if x inside $A_{e^{-u_n} - e^{-u_n^2}}(x_0)$, $\phi_n = 0$ if x is outside U_n , and decays to 0 at a linear rate on $S(n, x_0)$. So we have the estimate $\|\phi_n - \mathbf{1}_{\{X_0 > u_n\}}\|_1 < \mu(S(n, p))$ and $\|\phi_n\|_{Lip} \leq e^{-u_n^2}$. Also let $\psi_n = \mathbf{1}_{\{M_l \leq u_n\}}$. By ([8], Lemma 3.1), we then have

$$\left| \int \phi_n \psi_n \circ F^t d\mu - \int \phi_n d\mu \int \psi_n d\mu \right| \leq O(1)(\|\phi_n\|_\infty \tau_1^{\lfloor t/2 \rfloor} + \|\phi_n\|_{Lip} \alpha^{\lfloor t/2 \rfloor})$$

and

$$\begin{aligned} &|\mu(\{X_0 > u_n\} \cap \{M_{t,l} \leq u_n\}) - \mu(\{X_0 > u_n\})\mu(\{M_l \leq u_n\})| \\ &\leq \left| \int (\mathbf{1}_{\{X_0 > u_n\}} - \phi_n) \psi_n \circ F^t d\mu \right| + \left| \int \phi_n \psi_n \circ F^t d\mu - \int \phi_n d\mu \int \psi_n d\mu \right| \\ &\quad + \left| \int (\mathbf{1}_{\{X_0 > u_n\}} - \phi_n) d\mu \int \psi_n d\mu \right| \\ &\leq O(1)(n^{-2\delta v - \delta \log n} + \|\phi_n\|_\infty \tau_1^{\lfloor t/2 \rfloor} + \|\phi_n\|_{Lip} \alpha^{\lfloor t/2 \rfloor}) \end{aligned}$$

where τ_1 is from [8, Proposition 1.1]. Let $\gamma(n, t) = n^{-2\delta v - \delta \log n} + \|\phi_n\|_\infty \tau_1^{\lfloor t/2 \rfloor} + \|\phi_n\|_{Lip} \alpha^{\lfloor t/2 \rfloor}$, and $\|\phi_n\|_{Lip} \leq e^{-u_n^2} \leq O(1)n^{-2\tilde{\delta}v - \delta \log n}$ (by similar argument we did to get $\mu(S(n, x_0))$). Take $t = t_n = (\log n)^5$, so that $n\gamma(n, t_n) \rightarrow 0$ as $n \rightarrow \infty$. Thus $D_3(u_n)$ is established.

We now establish $D'(u_n)$. Note $F^{-j}(U_n) = \{X_j > u_n\}$, then

$$\mu(X_0 > u_n, X_j > u_n) = \mu(U_n \cap F^{-j}U_n) \leq \int_{\gamma \in \Gamma} m(\pi(U_n)_\gamma \cap T^{-j}(\pi(U_n)_\gamma)) d\nu(\gamma)$$

Where π is the projection map onto the first dimension. For each leaf $\gamma \in \Gamma$, define $B_{r_n, \gamma} = \pi(U_n)_\gamma = B(q_\gamma e^{-u_n}, \pi(x_0))$, where $r_n = e^{-u_n}$, $0 \leq q_\gamma \leq 1$.

Consider points where the local unstable manifold is less than r^3 , so that the integral splits as follows

$$\begin{aligned} \int_{\gamma \in \Gamma} m(\pi(U_n)_\gamma \cap T^{-j}(\pi(U_n)_\gamma)) d\nu(\gamma) &= \int_{\gamma \in \Gamma_1} m(\pi(U_n)_\gamma \cap T^{-j}(\pi(U_n)_\gamma)) d\nu(\gamma) \\ &\quad + \int_{\gamma \in \Gamma_2} m(\pi(U_n)_\gamma \cap T^{-j}(\pi(U_n)_\gamma)) d\nu(\gamma) \end{aligned}$$

where Γ_1 is the set of local unstable manifolds which has length less than r^3 , and $\Gamma_2 = \Omega/\Gamma_1$. The reason for doing so is because if the local unstable manifold has a short length, the point in the projection probably has no short return.

For $\gamma \in \Gamma_2$, define

$$E_{k, \gamma} = \{x \in B_{r_k, \gamma} : d(T^j x, x) < \frac{1}{k^{1/3}}, \text{ for some } 1 \leq j \leq (\log k)^5\}.$$

By [8, Proposition 4.2], there exists $0 < a < 1, 0 < \tilde{\theta} < 1$ such that

$$m(E_{k, \gamma}) < \tilde{\theta}^{(\log k^{1/3})^a}$$

We only need $a < 1/2$, so we take $a = 1/3$.

Let $0 < \beta \leq \frac{1}{2}$ and let $0 < \rho < 1$ such that $\rho\beta < \beta/3$.

Define the set

$$F_{k, \gamma} := \{m(B_{q_\gamma \exp(-k^\beta)}(x) \cap E_{\exp(k^{3\beta})}) \geq m(B_{q_\gamma \exp(-k^\beta)}(x)) \exp(-k^{\beta\rho})\}. \quad (3.1)$$

If $x \in F_{k, \gamma}$ then

$$\frac{m(B_{q_\gamma \exp(-k^\beta)}(x) \cap E_{\exp(k^{3\beta})})}{m(B_{q_\gamma \exp(-k^\beta)}(x))} \geq \exp(-k^{\beta\rho}); \quad (3.2)$$

If we define

$$M_l(x) := \sup_{r>0} \frac{1}{m(B_r(x))} \int_{B_r(x)} \mathbf{1}_{E_l}(y) dm(y)$$

we see immediately from the definition of $M_l(x)$ and (3.2) that for every $x \in F_{k, \gamma}$, $M_{e^{k^{3\beta}}}(x) \geq e^{-k^{\beta\rho}}$. Hence

$$F_{k, \gamma} \subset \{M_{e^{k^{3\beta}}}(x) \geq e^{-k^{\beta\rho}}\}. \quad (3.3)$$

A theorem of Hardy and Littlewood [12, Theorem 2.19] implies that

$$m(|M_l| > c) \leq \frac{\|\mathbf{1}_{E_l}\|_1}{c};$$

$\|\cdot\|_1$ is with respect to the one-dimensional Lebesgue measure. As $m(E_l) \leq \mathcal{O}(1)\tilde{\theta}^{(\log l^{1/3})^{1/3}}$ (recall $a = 1/3$),

$$m(F_{k,\gamma}) \leq \mathcal{O}(1)m(E_{\exp(k^{3\beta})})e^{k^{\beta\rho}} \leq \mathcal{O}(1)(e^{\alpha k^{\beta/3} + k^{\beta\rho}})$$

where $\alpha := \log \tilde{\theta}$ and k is large enough. Since $\beta/3 > \beta\rho$, $\sum_{k>0} m(F_{k,\gamma}) < \infty$. By the Borel Cantelli lemma, $m(\limsup F_{k,\gamma}) = 0$, and hence for m almost every x there exists an N_x such that for all $k \geq N_x$, $x \notin F_{k,\gamma}$ for each γ .

Let x_0 be such a generic point, and let N_{x_0} be the corresponding index beyond which x_0 does not belong to any $F_{k,\gamma}$. Since $\lim_{k \rightarrow \infty} e^{(k+1)^\beta} e^{-k^\beta} = 1$ the fact that we restricted to a subsequence is of no consequence, and we obtain the following estimate for all n sufficiently large. If $1 \leq j \leq (\log n)^5$, then

$$m(B_{r_n,\gamma} \cap T^{-j} B_{r_n,\gamma}) \leq m(B_{r_n,\gamma}) \exp(-u_n^\rho) \quad (3.4)$$

Summing over $1 \leq j \leq (\log n)^5$ and taking limits as $n \rightarrow \infty$ we obtain:

$$\begin{aligned} & n \sum_1^{(\log n)^5} \int_{\gamma \in \Gamma_2} m(\pi(U_n)_\gamma \cap T^{-j}(\pi(U_n)_\gamma)) d\nu(\gamma) \\ & \leq n \sum_1^{(\log n)^5} e^{-u_n^\rho} \int_{\gamma \in \Gamma_2} m(\pi(U_n)_\gamma) d\nu(\gamma) \\ & \leq n \sum_1^{(\log n)^5} e^{-u_n^\rho} \mu(U_n) \\ & = (\log n)^5 e^{-u_n^\rho} e^{-v} \rightarrow 0 \end{aligned}$$

since u_n has estimates in part(a) of Lemma 2.

And for Γ_1 ,

$$n \sum_1^{(\log n)^5} \int_{\gamma \in \Gamma_1} m(\pi(U_n)_\gamma \cap T^{-j}(\pi(U_n)_\gamma)) d\nu(\gamma) \leq n(\log n)^5 e^{-3u_n} \rightarrow 0$$

Consequently we have

$$n \sum_1^{(\log n)^5} \mu(X_0 > u_n, X_0 \circ F^j > u_n) \rightarrow 0$$

Finally, similarly to the argument in the case of Planar Dispersing Billiard Maps in [8, section 4.1.3], we use exponential decay of correlations to show

$$\lim_{n \rightarrow \infty} n \sum_{(\log n)^5}^{p=\sqrt{n}} \mu(X_0 > u_n, X_0 \circ F^j > u_n) = 0.$$

□

Now we can prove Theorem 1.7.

Proof of Theorem 1.7.

We follow the steps of proof of Theorem 1.6, and use the techniques from the proof of Theorem 1.8, we have:

$$\begin{aligned}
& \mu(A_i \cap F^{-(j-i)} A_j) \leq \mu(A_i \cap F^{-(j-i)} A_i) \\
& \sim \int_{\Gamma} m(\tilde{x} \in (A_i)_{\gamma} : F^{j-i}(\tilde{x}) \in A_i) d\nu(\gamma) \\
& \leq \int_{\Gamma} m(\pi(A_i)_{\gamma} \cap T^{-(j-i)}(\pi(A_i)_{\gamma})) d\nu(\gamma) \\
& = \int_{\Gamma_1} m(\pi(A_i)_{\gamma} \cap T^{-(j-i)}(\pi(A_i)_{\gamma})) d\nu(\gamma) \\
& \quad + \int_{\Gamma_2} m(\pi(A_i)_{\gamma} \cap T^{-(j-i)}(\pi(A_i)_{\gamma})) d\nu(\gamma)
\end{aligned}$$

where Γ_1 is the set of local unstable manifolds which has length less than r^3 , and $\Gamma_2 = \Omega/\Gamma_1$. Thus,

$$\int_{\Gamma_1} m(\pi(A_i)_{\gamma} \cap T^{-(j-i)}(\pi(A_i)_{\gamma})) d\nu(\gamma) \leq r_i^3 \leq \mu(A_i)^{\frac{3}{d+\eta}}$$

and by (3.4)

$$\int_{\Gamma_2} m(\pi(A_i)_{\gamma} \cap T^{-(j-i)}(\pi(A_i)_{\gamma})) d\nu(\gamma) \leq \mu(A_i) \exp(-u_i^{\rho})$$

Let $f_k = \mathbf{1}_{A_k} \circ F^k(x, y)$ and $E(f_k) = \mu(A_k)$, so that

$$\begin{aligned}
& \sum_{j=i+1}^n (E(f_i f_j) - E(f_i)E(f_j)) \\
& = \sum_{j=i+1}^n \mu(A_i \cap F^{j-i} A_j) - \mu(A_i)\mu(A_j) \\
& = \sum_{j=i+1}^{i+(\log i)^5} [\mu(A_i \cap F^{j-i} A_j) - \mu(A_i)\mu(A_j)] + \sum_{j>i+(\log i)^5} [\mu(A_i \cap F^{j-i} A_j) - \mu(A_i)\mu(A_j)] \\
& \leq \sum_{j=i+1}^{i+(\log i)^5} \left[\mu(A_i)^{\frac{3}{d+\eta}} + \mu(A_i) \exp(-u_i^{\rho}) \right] + C\mu(A_i) \\
& \leq (\log i)^5 \mu(A_i)^{\frac{3}{d+\eta}-1} \mu(A_i) + C_1 \mu(A_i) + C\mu(A_i) \leq \tilde{C} \mu(A_i)
\end{aligned}$$

where $0 < \rho < 1/3$. Therefore we have the (SP) property, and the Strong Borel Cantelli property then follows. □

4 Lorenz Flow

Let M be the Riemannian manifold, associated with Lorenz flows, endowed with a metric d_M , and $f_t : M \rightarrow M$ the Lorenz C^1 -flow. $\Omega \subset M$ is a transverse cross-section of the flow which is a C^1 -submanifold with boundary, as we stated in previous sections. We know $F : \Omega^* \rightarrow \Omega$ preserves a probability measure μ , where $\Omega = [-1/2, 1/2] \times [-1/2, 1/2]$ and $\Omega^* = ([-1/2, 1/2] \setminus \{0\}) \times [-1/2, 1/2]$. Let $h : \Omega \rightarrow \mathbb{R}_+$ be the first return time of the flow to Ω , and $h \notin L^1(\mu)$. Consider the suspension space

$$\Omega^h = \{(p, u) \in \Omega \times \mathbb{R} \mid 0 \leq u \leq h(p)\} / \sim, \quad \text{where } (p, h(p)) \sim (F(p), 0)$$

We model the flow $f_t : M \rightarrow M$ in the standard way by the suspension flow $\tilde{f}_t : \Omega^h \rightarrow \Omega^h$, $\tilde{f}_t(p, u) = (p, u + t) / \sim$. Denote the metric on Ω by d_Ω , and we define a metric d_{Ω^h} on Ω^h by

$$d_{\Omega^h}((p, u), (q, v)) = \sqrt{d_\Omega(p, q)^2 + |u - v|^2}$$

Then we can introduce a projection map $\pi_M : \Omega^h \rightarrow M$, $(p, t) \mapsto f_t(p)$, which is a local C^1 -diffeomorphism. μ is an invariant ergodic probability measure for the first return map, i.e. our Lorenz map, $F : \Omega^* \rightarrow \Omega$. This induces (in the standard way) an invariant measure μ^h , on the suspension Ω^h , which is given by $d\mu \times dm / \bar{h}$ and $\bar{h} = \int_\Omega h d\mu$. Then μ^h determines a f_t -invariant measure μ_M on M by $\mu_M(A) = \mu^h(\pi_M^{-1}A)$ for measurable sets A .

Consider a measurable observation $\varphi : \Omega^h \rightarrow \mathbb{R}$ such that $\varphi(x) = -\log d_{\Omega^h}(x, x_0)$ where x_0 is any point in Ω^h , then φ has a logarithmic singularity at x_0 . Define $\Phi : \Omega \rightarrow \mathbb{R}$ by

$$\Phi(p) := \max\{\varphi(f_s(p)) \mid 0 \leq s < h(p)\}$$

Denote

$$\varphi_t(p) := \max\{\varphi(f_s(p)) \mid 0 \leq s < t\}$$

$$\Phi_N(p) := \max\{\Phi(F^k(p)) \mid 0 \leq k < N\}$$

Then we have our main theorem in the flow case:

Theorem 4.1. *Assume that F is the Lorenz map and f_t is the corresponding Lorenz flow. Assume the levels $\{u_n\}$ satisfy*

$$n\mu(\Phi_0 > u_n) \rightarrow e^{-v}$$

which gives some normalizing constants $a_n > 0$ and b_n such that

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} a_n |b_{[n+\epsilon n]} - b_n| = 0$$

$$\lim_{\epsilon \rightarrow 0} \limsup_{n \rightarrow \infty} \left| 1 - \frac{a_{[n+\epsilon n]}}{a_n} \right| = 0$$

Then Φ_N satisfies a Type I extreme value law, i.e.

$$a_N(\Phi_N - b_N) \rightarrow_d e^{-e^{-v}}$$

implies that φ_t also satisfies a Type I extreme value law,

$$a_{\lfloor T/\bar{h} \rfloor}(\varphi_T - b_{\lfloor T/\bar{h} \rfloor}) \rightarrow_d e^{-e^{-v}}$$

Proof. It is a consequence of [10, Theorem 2.6] and Theorem 1.8. [10, Sublemma 4.18] takes care of the issue that h is not bounded. \square

5 Appendices

5.1 Gal-Koksma Theorem.

We recall the following result of Gal and Koksma as formulated by W. Schmidt [13, 14] and stated by Sprindzuk [15]:

Let $(\Omega, \mathcal{B}, \mu)$ be a probability space and let $f_k(\omega)$, $(k = 1, 2, \dots)$ be a sequence of non-negative μ measurable functions and g_k, h_k be sequences of real numbers such that $0 \leq g_k \leq h_k \leq 1$, $(k = 1, 2, \dots)$. Suppose there exists $C > 0$ such that

$$\int \left(\sum_{m < k \leq n} (f_k(\omega) - g_k) \right)^2 d\mu \leq C \sum_{m < k \leq n} h_k \quad (*)$$

for arbitrary integers $m < n$. Then for any $\epsilon > 0$

$$\sum_{1 \leq k \leq n} f_k(\omega) = \sum_{1 \leq k \leq n} g_k + O(\Theta^{1/2}(n) \log^{3/2+\epsilon} \Theta(n))$$

for μ a.e. $\omega \in \Omega$, where $\Theta(n) = \sum_{1 \leq k \leq n} h_k$.

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